ECE 307 – Techniques for Engineering Decisions

10. Basic Probability Review

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OUTLINE

- Definitions
- Axioms on probability
- Conditional probability
- Independence of events
- Probability distributions and densities
  - discrete
  - continuous

SAMPLE SPACE

- Consider an experiment with uncertain outcomes but with the entire set of all possible outcomes known
- The *sample space* S is the set of all possible outcomes, i.e., every outcome is an element of S

SAMPLE SPACE

- **Examples of sample spaces**
  - **flipping a coin:** $\mathscr{S} = \{ H, T \}$
  - **tossing a die:** $\mathscr{S} = \{ 1, 2, 3, 4, 5, 6 \}$
  - **flipping two coins:** $\mathscr{S} = \{ (H, H), (H, T), (T, H), (T, T) \}$
  - **tossing two dice:** $\mathscr{S} = \{ (i, j) : i, j = 1, \ldots, 6 \}$
  - **hours of life of a device:** $\mathscr{S} = \{ x : 0 \leq x < \infty \}$

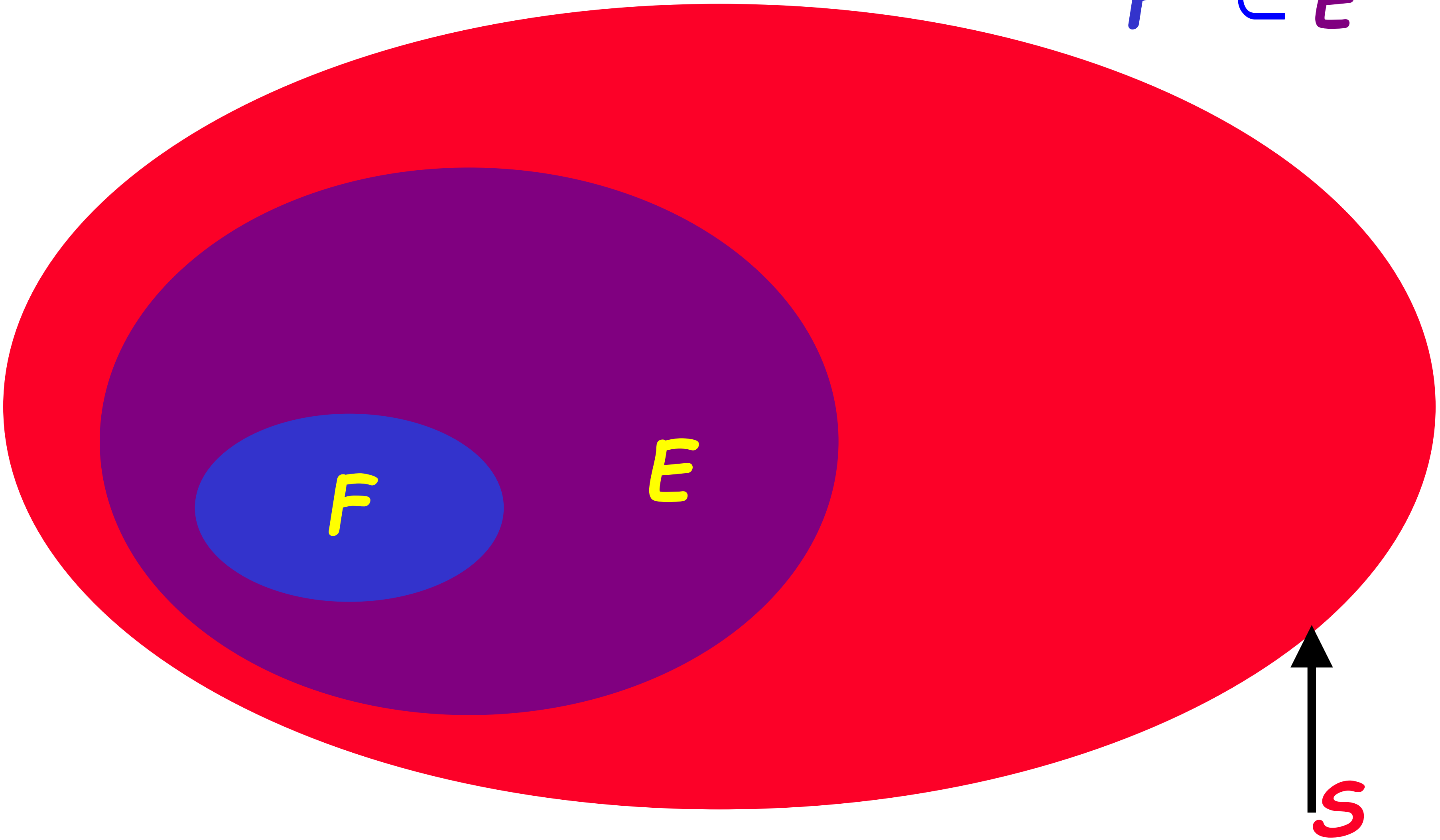
SUBSETS

- **We say a set E is a subset of a set F if E is contained in F and we write $E \subset F$ or $F \supset E$**
- **If E and F are sets of events, then $E \subset F$ implies that each event in E is also an event in F**
- **Theorem**

$$E \subset F \text{ and } F \supset E \Leftrightarrow E = F$$

SUBSETS

$F \subset E$



EVENTS

- An *event* E is an element or a subset of the *sample space* S
- Some examples of events are:
  - flipping a coin: $\mathscr{E} = \{ H \}, \ \mathscr{F} = \{ T \}$
  - tossing a die: $\mathscr{E} = \{ 2, 4, 6 \}$ is the event that the die lands on an even number

EVENTS

- **flipping two coins:** $\mathscr{E} = \left\{ \left(H, H \right), \left(H, T \right) \right\}$ **is the event of the outcome *H* on the first coin**
- **tossing two dice:**

$$\mathscr{E} = \left\{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \right\}$$

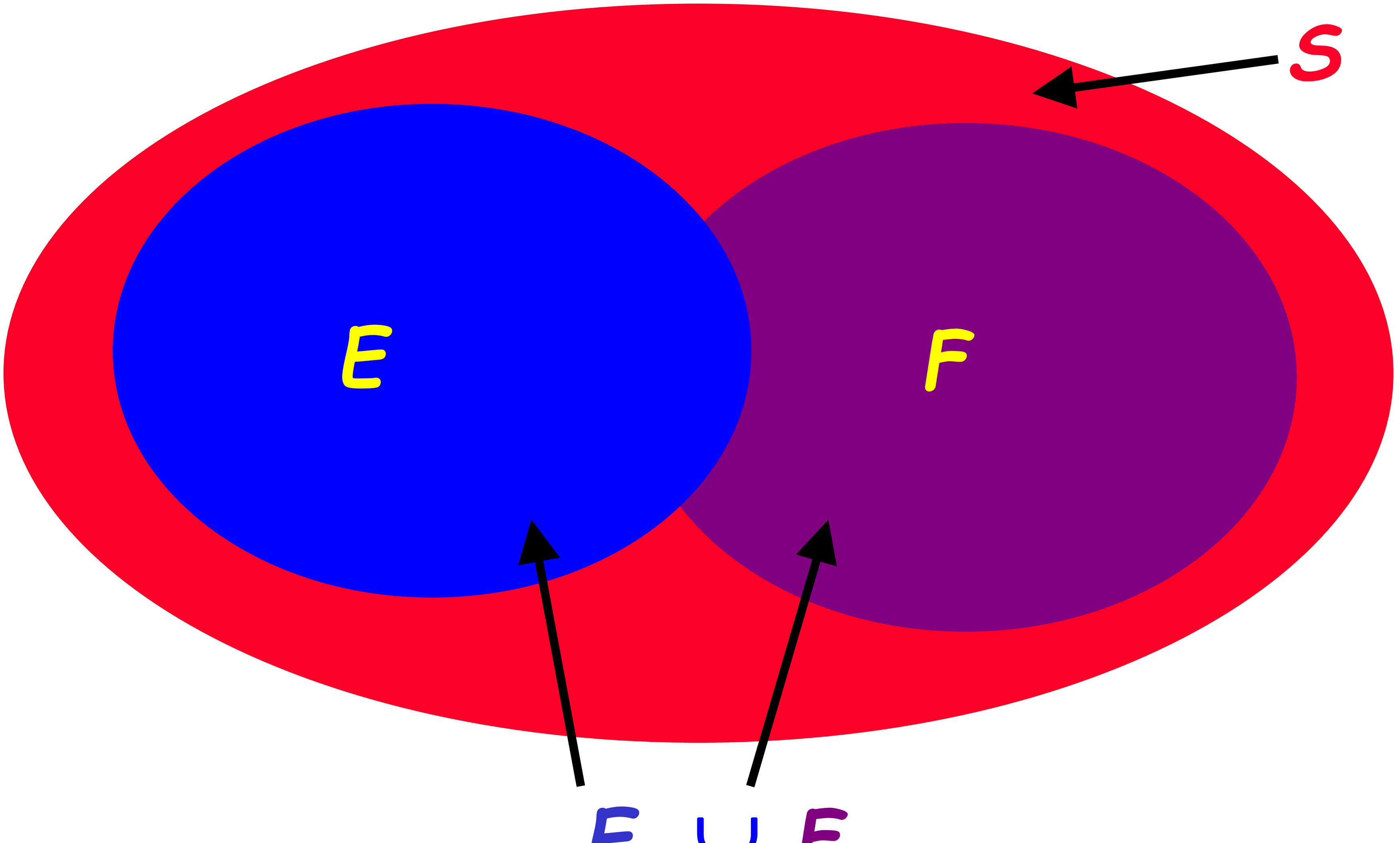
is the event of sum of the two tosses is 7

- **hours of life of a device:** $\mathscr{E} = \left\{ 5 < x \leq 10 \right\}$ **is the event that the life of a device is greater than 5 and at most 10 *hours***

UNION OF SUBSETS

- We consider two subsets E and F ; the *union* of E and F is denoted by $E \cup F$ and is the set of all the elements that are either in E or in F or in both E and F
- E and F represent subsets of events, the $E \cup F$ occurs only if either E or F or both occur
- $E \cup F$ is equivalent to the logical *or*

UNION OF SUBSETS



UNION OF SUBSETS

- **Examples:**
  - $\mathscr{E} = \{2, 4, 6\}, \mathscr{F} = \{1, 2, 3\} \Rightarrow \mathscr{E}$ » $\mathscr{F} = \{1, 2, 3, 4, 6\}$
  - $\mathscr{E} = \{H\}, \mathscr{F} = \{T\} \Rightarrow \mathscr{E} \cup \mathscr{F} = \{H, T\} = \mathscr{S}$
  - ***E*** **= set of outcomes of tossing two dice with sum being an even number**

    F **= set of outcomes of tossing two dice with sum being an odd number**

$$\Longrightarrow \quad \mathscr{E} \cup \mathscr{F} = \mathscr{S}$$

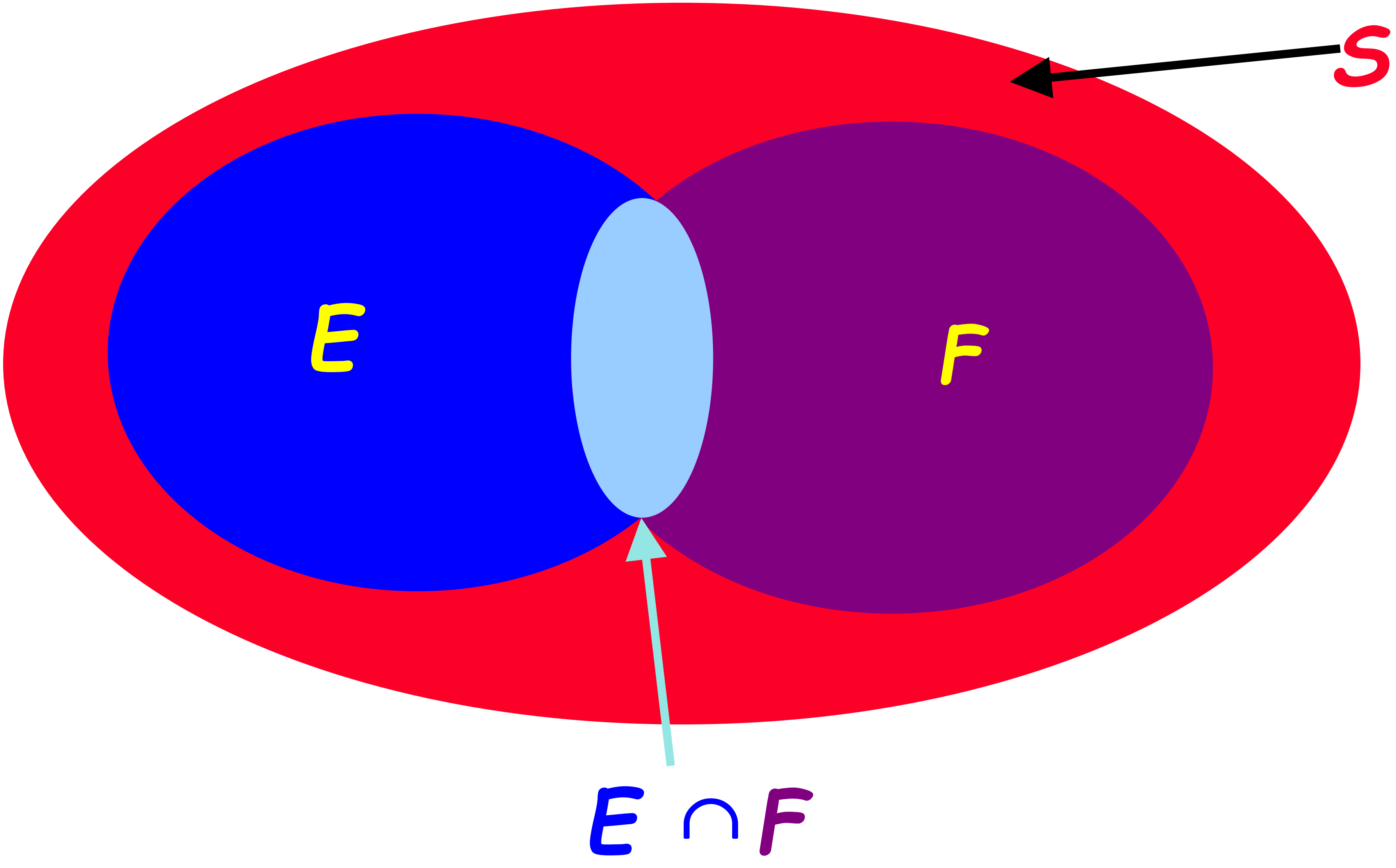
INTERSECTION OF SUBSETS

- We consider two subsets E and F ; the intersection of E and F , denoted by $E \cap F$, is the set of all the elements that are both in E *and* in F
- E and F represent subsets of events, then the events in $E \cap F$ occur only if both E and F occu
- $E \cap F$ is equivalent to the logical *and*

INTERSECTION OF SUBSETS

- We define $\varnothing$ to be the *empty* set, i.e., the set consisting of no elements
- For event subspaces E and F, if $E \cap F = \varnothing$ if and only if E and F are *mutually exclusive* events
- Examples:
  - $\mathscr{E} = \{H\}, \mathscr{F} = \{T\} \Rightarrow \mathscr{E} \cap \mathscr{F} = \varnothing$
  - $\mathscr{E} = \{1,3,5\}, \mathscr{F} = \{1,2,3\} \Rightarrow \mathscr{E} \cap \mathscr{F} = \{1,3\}$

INTERSECTION OF SUBSETS



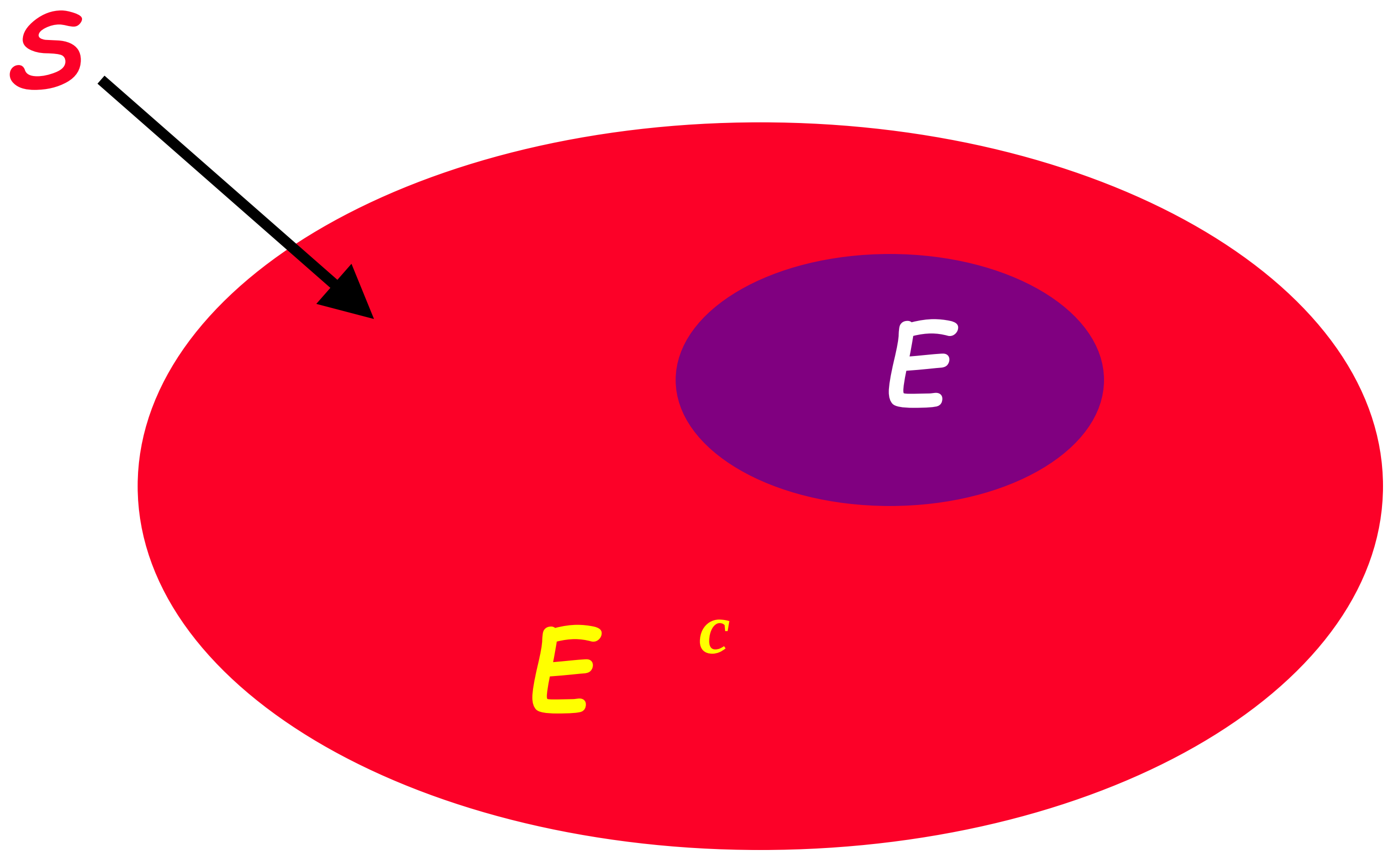
GENERALIZATION OF CONCEPTS

- We consider the countable subsets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots$ in the state space $\mathcal{S}$
- The term $\bigcup_i \mathcal{E}_i$ is defined to be that subset consisting of those elements that are in $\mathcal{E}_i$ for *at least one value* of $i = 1, 2, \ldots$
- The term $\bigcap_i \mathcal{E}_i$ is defined to be the subset consisting of those elements that are *in every subset* $\mathcal{E}_i$, $i = 1, 2, \ldots$

COMPLEMENT OF A SUBSET

- The complement E^c of a set E is the set of all elements in the sample space S not in E
- By definition, $S^c = \varnothing$
- For the example of tossing two dice, the subset $\mathscr{E} = \{(1,6),\ (2,5),\ (3,4),\ (4,3),\ (5,2),\ (6,1)\}$ is the collection of events that the sum of dice is 7; then, E^c is the collection of events that the sum of dice is not 7

COMPLEMENT OF A SUBSET



DE MORGAN'S LAWS

- De Morgan's laws establish some important relationships between $\bigcup$, $\bigcap$ and ^{c}
- The first De Morgan law states:

$$\left(\bigcup_{i=1}^{n} \mathscr{E}_i \right)^c = \bigcap_{i=1}^{n} \mathscr{E}_i^{\,c}$$

- The second De Morgan law states:

$$\left(\bigcap_{i=1}^{n} \mathscr{E}_i \right)^c = \bigcup_{i=1}^{n} \mathscr{E}_i^{\,c}$$

DEFINITION OF PROBABILITY

- Consider an event $\mathcal{E}$ in the sample space $\mathcal{S}$ and us denote by $n(\mathcal{E})$ the number of times that the event $\mathcal{E}$ occurs in a total of n random draws
- We define the *probability* $P\{\mathcal{E}\}$ for the sample space of the event $\mathcal{E}$ by

$$P\{\mathscr{E}\} = \lim_{n \to \infty} \frac{n(\mathscr{E})}{n}$$

PROBABILITY AXIOMS

- **Axiom 1:**

$$0 \leq P\left\{\mathscr{E}\right\} \leq 1$$

the probability that the outcome of the experiment is the event $\mathscr{E}$ **lies in** $[0, 1]$

- **Axiom 2:**

$$P\left\{\mathscr{S}\right\} = 1$$

the probability associated with all the events in the sample space $\mathscr{S}$ **is 1 as** $\mathscr{S}$ **is the collection of all the events of the sample space**

PROBABILITY AXIOMS

- **Axiom 3: For any collection of mutually exclusive events $E_1, E_2, \ldots$ with $E_i \cap E_j = \emptyset, i \neq j$,**

$$P\left\{\bigcup_i \mathscr{E}_i\right\} = \sum_i P\{\mathscr{E}_i\},$$

i.e., for a collection of mutually exclusive events, the probability that at least one of the events of the collection occurs is the sum of the

APPLICATIONS OF THE AXIOMS

- In a coin tossing experiment, we assume that a head is equally likely to appear as a tail so that:

$$P\{\{H\}\} = P\{\{T\}\} = 0.5$$

- If the coin is biased and we have the situation that the head is twice as likely to appear as the tail, then

$$P\{\{H\}\} = 0.66\dot{6} \quad \text{and} \quad P\{\{T\}\} = 0.33\dot{3}$$

EXAMPLE

- In a die tossing experiment, we assume that each of the six sides is equally likely to appear so that

$$P\{\{1\}\} = P\{\{2\}\} = P\{\{3\}\} = P\{\{4\}\} = P\{\{5\}\} = P\{\{6\}\} = 0.16\dot{6}$$

- The probability of the event that the toss results in an even number is:

$$P\{\{2,4,6\}\} = P\{\{2\}\} + P\{\{4\}\} + P\{\{6\}\} = \left(0.16\dot{6}\right)3 = 0.5$$

SIMPLE PROBABILITY THEOREMS

- The theorem on a complementary set states that the probability of the complement of the event *E* is 1 minus the probability the event itself

$$P\left\{\mathscr{E}^{c}\right\} = 1 - P\left\{\mathscr{E}\right\}$$

- For example, if the probability of obtaining an outcome $\left\{H\right\}$ on a biased coin is 0.375, then the probability of obtaining an outcome $\left\{T\right\}$ is 0.625

SIMPLE PROBABILITY THEOREMS

- The theorem on a subset considers two subsets ***E*** and ***F*** of ***S*** and states

$$\mathscr{E} \subset \mathscr{F} \Rightarrow P\left\{\mathscr{E}\right\} \leq P\left\{\mathscr{F}\right\}$$

- For example, the probability of rolling a 1 with a die is less than or equal to the probability of rolling an odd value with that same die
- Theorem on the union of two subsets concerns two subsets ***E*** and ***F*** of ***S*** and states that

$$P\left\{\mathscr{E} \cup \mathscr{F}\right\} = P\left\{\mathscr{E}\right\} + P\left\{\mathscr{F}\right\} - P\left\{\mathscr{E} \cup \mathscr{F}\right\}$$

SIMPLE PROBABILITY THEOREMS

- For example, in the experiment of tossing two fair coins

$$\mathscr{S} = \{\{H,H\},\{H,T\},\{T,H\},\{T,T\}\}$$

and the four outcomes are equally likely; the subset of the events that either the first or the second coin falls on H is the union of the subsets of events

$$\mathscr{E} = \{\{H,H\},\{H,T\}\}$$

that the first coin is H and the subset of events

SIMPLE PROBABILITY THEOREMS

$$\mathscr{F} = \big\{ \{H,H\}, \{T,H\} \big\}$$

represents the event second coin toss is *H*; so

$$P\{\mathscr{E} \cup \mathscr{F}\} = P\{\mathscr{E}\} + P\{\mathscr{F}\} - P\{\mathscr{E} \cap \mathscr{F}\}$$

$$= 0.5 + 0.5 - \underbrace{P\big\{ \{H,H\} \big\}}_{0.25}$$

$$= 0.75$$

CONDITIONAL PROBABILITY

- **A conditional event *E* is one that occurs given that some other event *F* has already occurred**
- **The *conditional probability* $P\{\mathscr{E} \mid \mathscr{F}\}$ is the probability that event *E* occurs given that event *F* has occurred and is defined by**

$$P\{\mathscr{E} \mid \mathscr{F}\} = \frac{P\{\mathscr{E} \cap \mathscr{F}\}}{P\{\mathscr{F}\}}$$

CONDITIONAL PROBABILITY

- As an example, consider that a coin is flipped twice and assume that each of the events in

$$\mathscr{S}=\left\{\left\{H,H\right\},\left\{H,T\right\},\left\{T,H\right\},\left\{T,T\right\}\right\}$$

is equally likely to occur; then, $\left\{H\right\}$ and $\left\{T\right\}$ are equally likely to occur

- The conditional probability that both flips result in $\left\{H\right\}$, given that the first flip is $\left\{H\right\}$ is obtained as follows:

CONDITIONAL PROBABILITY

$$\mathscr{E} = \{\{H,H\}\}$$

$$\mathscr{F} = \{\{H,H\},\{H,T\}\}$$

$$P\{\mathscr{E} \mid \mathscr{F}\} = \frac{P\{\mathscr{E} \cap \mathscr{F}\}}{P\{\mathscr{F}\}} = \frac{\overbrace{P\{\{H,H\}\}}^{0.25}}{\underbrace{P\{\{H,H\},\{H,T\}\}}_{0.5}} = 0.5$$

CONDITIONAL PROBABILITY APPLICATION

- **Bev must decide whether to select either a *French* or a *Chemistry* course**
- **She estimates to have probability of 0.5 to get an *A* in a *French* course and that of 0.333 in a *Chemistry* course, which she actually prefers**
- **She decides by flipping a fair coin and determines the probability she can get *A* in *Chemistry*:**

CONDITIONAL PROBABILITY APPLICATION

- $\mathcal{C}$ is the event that she takes *Chemistry*

- $\mathcal{A}$ is the event that she receives an *A* in whichever course she takes

- then $P\{\mathscr{C} \cap \mathscr{A}\}$ is the probability she gets *A* in *Chemistry*

$$P\{\mathscr{C} \cap \mathscr{A}\} = P\{\mathscr{C}\}\ P\{\mathscr{A} | \mathscr{C}\} = (0.5)\,(0.333) = 0.166$$

BAYES' THEOREM

- Consider two subsets of events E and F in S ; then,

$$P\{\mathscr{E} \mid \mathscr{F}\} = \frac{P\{\mathscr{F} \mid \mathscr{E}\} P\{\mathscr{E}\}}{P\{\mathscr{F} \mid \mathscr{E}\} P\{\mathscr{E}\} + P\{\mathscr{F} \mid \mathscr{E}^{c}\} P\{\mathscr{E}^{c}\}}$$

- The proof of this theorem makes use of the definition of conditional probability

$$P\{\mathscr{E} | \mathscr{F}\} = \frac{P\{\mathscr{E} \cap \mathscr{F}\}}{P\{\mathscr{F}\}} = \frac{P\{\mathscr{F} | \mathscr{E}\} P\{\mathscr{E}\}}{P\{\mathscr{F}\}}$$

BAYES' THEOREM

and of the fact that any subset *F* is the union of two nonintersecting subsets

$$\mathscr{F} = \left\{ \mathscr{F} \cap \mathscr{E} \right\} \cup \left\{ \mathscr{F} \cap \mathscr{E}^{c} \right\}$$

- These expressions result from the relation

$$P\left\{ \bigcup_{i} \mathscr{E}_{i} \right\} = \sum_{i} P\left\{ \mathscr{E}_{i} \right\},$$

APPLICATION OF BAYES' THEOREM TO DIAGNOSIS

- A laboratory test is $95\,\%$ effective in correctly detecting a certain disease when it is present, but the test yields a false positive result for $1\,\%$ of the healthy persons tested, i.e., with probability 0.01, the test result incorrectly concludes that a healthy person has the disease
- We are given that $0.5\,\%$ of the population actually has the disease

APPLICATION OF BAYES' THEOREM TO DIAGNOSIS

- We compute the probability that a person has the disease given that his test result is positive
- D is the event that the tested person actually has the disease and

$$P\{D\} = 0.005$$

- E is the event that the test result is positive

A DIAGNOSIS EXAMPLE COMPUTATION

- We evaluate the

$$P\left\{ \mathscr{D} \,\middle|\, \mathscr{E} \right\} = \frac{P\left\{ \mathscr{E} \,\middle|\, \mathscr{D} \right\} P\left\{ \mathscr{D} \right\}}{P\left\{ \mathscr{E} \,\middle|\, \mathscr{D} \right\} P\left\{ \mathscr{D} \right\} + P\left\{ \mathscr{E} \,\middle|\, \mathscr{D}^{c} \right\} P\left\{ \mathscr{D}^{c} \right\}}$$

$$= \frac{(0.95) \cdot (0.005)}{(0.95) \cdot (0.005) + (0.01) \cdot (0.995)}$$

$$= 0.323$$

MULTIPLE CHOICE EXAM APPLICATION

- In answering a question on a multiple choice test, a student either knows the answer or he guesses: the probability is p that the student knows the answer and so $(1-p)$ is the probability that he guesses; a student who guesses has a probability of $1/m$ to be correct where m is the number of multiple choice alternatives

MULTIPLE CHOICE EXAM APPLICATION

- We wish to compute the conditional probability that a student knows the answer to a question which he answered correctly
- To evaluate we define
  - C is the event that the student answers the question correctly
  - K is the event that he actually knows the answer with $P\{K\} = p$

MULTIPLE CHOICE EXAM APPLICATION

$$P\{\mathscr{K} \mid \mathscr{C}\} = \frac{P\{\mathscr{K} \cap \mathscr{C}\}}{P\{\mathscr{C}\}}$$

$$= \frac{P\{\mathscr{C} \mid \mathscr{K}\} P\{\mathscr{K}\}}{P\{\mathscr{C} \mid \mathscr{K}\} P\{\mathscr{K}\} + P\{\mathscr{C} \mid \mathscr{K}^c\} P\{\mathscr{K}^c\}}$$

$$= \frac{(1)(p)}{(1)(p) + \left[(1/m)(1-p)\right]} = \frac{mp}{1+(m-1)p}$$

- If $m = 5$ and $p = 0.5$, the probability that a student knew the answer to a question he correctly answered is 5/6

CONDITIONAL PROBABILITY GENERALIZATION

- Consider three events A, B and C in the sample space S
- We apply the conditional probability definition repeatedly to evaluate $P\{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}\}$

$$P\{\mathscr{A} \cap \mathscr{B} \cap \mathscr{C}\} = P\{\mathscr{A} \mid \mathscr{B} \cap \mathscr{C}\} \cdot P\{\mathscr{B} \cap \mathscr{C}\}$$

$$= P\{\mathscr{A} \mid \mathscr{B} \cap \mathscr{C}\} \cdot P\{\mathscr{B} \mid \mathscr{C}\} \cdot P\{\mathscr{C}\}$$

CONDITIONAL PROBABILITY GENERALIZATION

- **However, we also have that**

$$P\left\{\mathscr{A}\cap\mathscr{B}\,\middle|\,\mathscr{C}\right\}\cdot P\left\{\mathscr{C}\right\} = P\left\{\mathscr{A}\cap\mathscr{B}\cap\mathscr{C}\right\}$$

$$= P\left\{\mathscr{A}\,\middle|\,\mathscr{B}\cap\mathscr{C}\right\}P\left\{\mathscr{B}\,\middle|\,\mathscr{C}\right\}\cdot P\left\{\mathscr{C}\right\}$$

and therefore

$$P\left\{\mathscr{A}\cap\mathscr{B}\middle|\mathscr{C}\right\} = P\left\{\mathscr{A}\,\middle|\,\mathscr{B}\cap\mathscr{C}\right\}\cdot P\left\{\mathscr{B}\middle|\mathscr{C}\right\}$$

INDEPENDENT EVENTS

- Two events *E* and *F* are said to be independent if and only if:

$$P\left\{\mathscr{E} \cap \mathscr{F}\right\} = \left[P\left\{\mathscr{E}\right\}\right]\left[P\left\{\mathscr{F}\right\}\right]$$

- Equivalently, *E* and *F* are independent if and only if:

$$P\left\{\mathscr{E} \middle| \mathscr{F}\right\} = P\left\{\mathscr{E}\right\}$$

- We give an example concerning picking cards from an ordinary deck of 52 playing cards

INDEPENDENT EVENTS

- *E* is the event that the selected card is an ace
- *F* is the event that the selected card is a spade
- *E* and *F* are independent since

$$P\{\mathscr{E} \cap \mathscr{F}\} = \frac{1}{52} \text{ and so } P\{\mathscr{E}\} = \frac{4}{52} \text{ and } P\{\mathscr{F}\} = \frac{13}{52}$$

INDEPENDENT EVENTS

- Two coins are flipped and all 4 distinct outcomes are assumed to be equally likely
- $\mathcal{E}$ is the event that the first coin is H and $\mathcal{F}$ is the event that the second coin is T
- Then, $\mathcal{E}$ and $\mathcal{F}$ are independent events with

$$P\{\mathscr{E}\} = P\{\{H,H\},\{H,T\}\} = 0.5$$

$$P\{\mathscr{F}\} = P\{\{H,T\},\{T,T\}\} = 0.5$$

and

$$P\{\mathscr{E} \cap \mathscr{F}\} = P\{\{H,T\}\} = (0.5)(0.5) = 0.25$$

PROBABILITY DISTRIBUTIONS

- A *probability distribution* describes mathematically the set of probabilities associated with each possible outcome of a random variable (*r.v.*)
- A *discrete probability distribution* is a distribution characterized by a random variable that can assume a *finite* set of possible values
- A *continuous probability distribution* is a distribution characterized by a random variable that can assume infinitely many values

DISCRETE PROBABILITY DISTRIBUTIONS

- ***Discrete probability distribution specification***: the probability distribution of a discrete *r.v.* $\underset{\sim}{Y}$ with *n* discrete possible values may be expressed in terms of either a
  - a ***probability mass function*** that provides the list of the probabilities for each possible outcome

DISCRETE PROBABILITY DISTRIBUTIONS

$$P\{\ \underset{\sim}{Y} = y_i\},\quad i = 1, 2, \ldots, n\ ;$$

or,

- a ***cumulative distribution function (c.d.f.)*** that gives the probability that a *r.v.* is less than or equal to a specific value

$$P\{\ \underset{\sim}{Y} \leq y_i\},\quad i = 1, 2, \ldots, n$$

DISCRETE PROBABILITY DISTRIBUTIONS

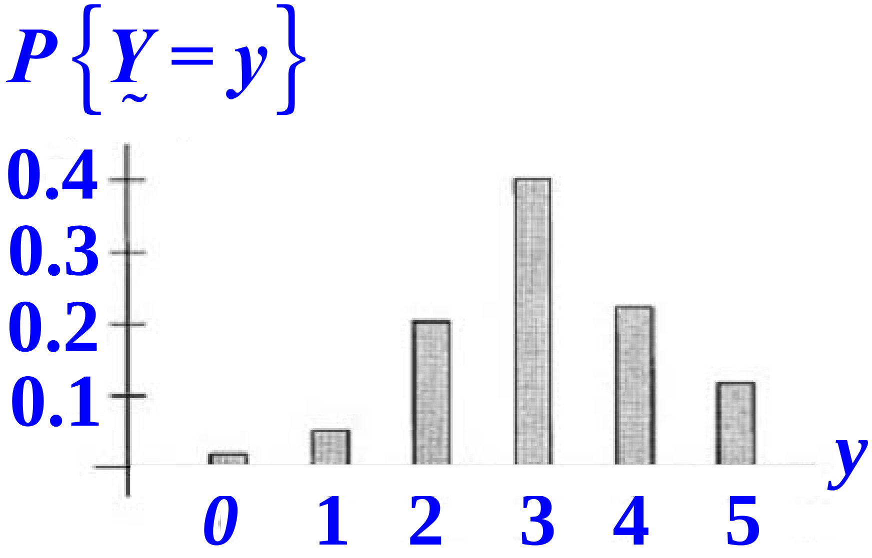
- As an example consider a set of raisin cookies with at most 5 raisins
- Assume that the probability that one of them has *0*, 1, 2, 3, 4 or 5 raisins is 0.02, 0.05, 0.2, 0.4, 0.22, and 0.11, respectively
- The *probability mass function* of the *r.v.* $\underset{\sim}{Y}$, defined to be the random number of raisins on a cookie, can be given either in tableau format or as a graph

DISCRETE PROBABILITY DISTRIBUTIONS

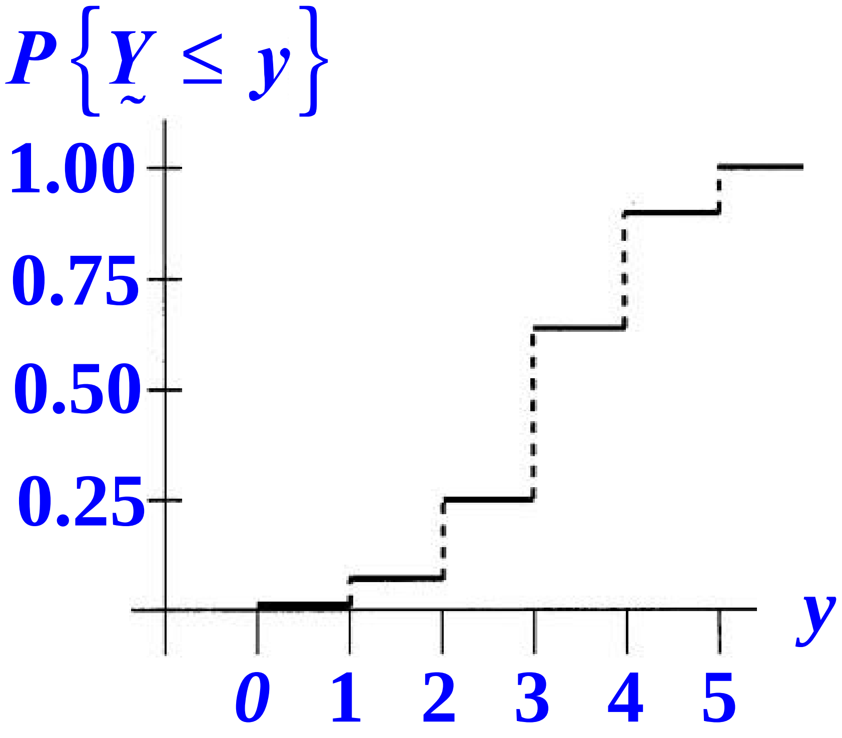
y	*probability mass function* $P\{\underset{\sim}{Y} = y\}$	*cumulative distribution function (c.d.f.)* $P\{\underset{\sim}{Y} \leq y\}$
0	0.02	0.02
1	0.05	0.07
2	0.20	0.27
3	0.4	0.67
4	0.22	0.89
5	0.11	1.00

DISCRETE PROBABILITY DISTRIBUTIONS

probability mass function



cumulative distribution function (c.d.f.)



THE EXPECTED VALUE

- The expected value $E\{\underset{\sim}{X}\}$ of the random variable $\underset{\sim}{X}$ is the probability–weighted average of all its possible values: for the set of possible values $\{x_1, x_2, \ldots, x_n\}$ for the variable $\underset{\sim}{X}$

$$\mu_{\underset{\sim}{X}} = E\{\underset{\sim}{X}\} = \sum_{i=1}^{n} x_i \, P\{\underset{\sim}{X} = x_i\}$$

- The expectation operator $E\{\cdot\}$ is also defined for any function $f(\cdot)$ of the *r.v.* $\underset{\sim}{X}$

THE EXPECTED VALUE

- Let

$$\underset{\sim}{Y} = f\left(\underset{\sim}{X}\right)$$

then

$$E\left\{\underset{\sim}{Y}\right\} = E\left\{f\left(\underset{\sim}{X}\right)\right\}$$

- In general ,

$$E\left\{f\left(\underset{\sim}{X}\right)\right\} \neq f\left(E\left\{\underset{\sim}{X}\right\}\right)$$

THE EXPECTED VALUE

- If $f\{\underset{\sim}{X}\}$ is affine, then,

$$E\left\{f\left(\underset{\sim}{X}\right)\right\} = f\left(E\left\{\underset{\sim}{X}\right\}\right)$$

  - $\underset{\sim}{Y} = a + b\,\underset{\sim}{X}$

    then

$$E\left\{\underset{\sim}{Y}\right\} = a + bE\left\{\underset{\sim}{X}\right\}$$

  - $\underset{\sim}{Y} = \underset{\sim}{X}_1 + \ldots + \underset{\sim}{X}_n$

    then

$$E\left\{\underset{\sim}{Y}\right\} = E\left\{\underset{\sim}{X}_1\right\} + \ldots + E\left\{\underset{\sim}{X}_n\right\}$$

THE VARIANCE

- The *variance* $var\{\underset{\sim}{X}\}$ of the random variable $\underset{\sim}{X}$ is the expected value of the squared difference between the uncertain quantities and their expected value $E\{\underset{\sim}{X}\}$:

$$var\{\underset{\sim}{X}\} \triangleq E\left\{\left[\underset{\sim}{X} - E\{\underset{\sim}{X}\}\right]^2\right\} = \sum_{i=1}^{n}\left(x_i - \mu_{\underset{\sim}{X}}\right)^2 P\{\underset{\sim}{X} = x_i\}$$

THE VARIANCE

- for $\underset{\sim}{Y} = a + b\underset{\sim}{X}$

$$var\left\{\underset{\sim}{Y}\right\} = var\left\{a + b\underset{\sim}{X}\right\}$$

$$= E\left\{\left[\left(a + b\underset{\sim}{X}\right) - \left(a + bE\left\{\underset{\sim}{X}\right\}\right)\right]^2\right\}$$

$$= E\left\{\left[b\underset{\sim}{X} - bE\left\{\underset{\sim}{X}\right\}\right]^2\right\}$$

$$= \left(b^2\right)\underbrace{E\left\{\left[\underset{\sim}{X} - E\left\{\underset{\sim}{X}\right\}\right]^2\right\}}_{var\left\{\underset{\sim}{X}\right\}}$$

$$= \left(b^2\right)var\left\{\underset{\sim}{X}\right\}$$

THE VARIANCE

- for

$$\underset{\sim}{Y} = \underset{\sim}{X}_1 + \ \ldots \ + \underset{\sim}{X}_n \text{ and } P\left\{ \underset{\sim}{X}_i \,\middle|\, \underset{\sim}{X}_j \right\} = P\left\{ \underset{\sim}{X}_i \right\} \ \forall \ i \neq j$$

then

$$var\left\{ \underset{\sim}{Y} \right\} = var\left\{ \underset{\sim}{X}_1 \right\} + \ \ldots \ + var\left\{ \underset{\sim}{X}_n \right\}$$

- The standard deviation $\sigma_{\underset{\sim}{X}}$ is given by

$$\sigma_{\underset{\sim}{X}} = \sqrt{var\left\{ \underset{\sim}{X} \right\}}$$

COVARIANCE AND CORRELATION COEFFICIENT

- The *covariance* $cov\{\underset{\sim}{X}, \underset{\sim}{Y}\}$ is defined by

$$cov\{\underset{\sim}{X},\underset{\sim}{Y}\} \triangleq E\Big\{\big(\underset{\sim}{X} - E\{\underset{\sim}{X}\}\big)\big(\underset{\sim}{Y} - E\{\underset{\sim}{Y}\}\big)\Big\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\big[x_i - E\{\underset{\sim}{X}\}\big]\big[y_j - E\{\underset{\sim}{Y}\}\big] P\{\underset{\sim}{X} = x_i, \underset{\sim}{Y} = y_j\}$$

- The *correlation* $\rho_{\underset{\sim}{X}\underset{\sim}{Y}}$ is defined by

$$\rho_{\underset{\sim}{X}\underset{\sim}{Y}} = \frac{cov\{\underset{\sim}{X},\underset{\sim}{Y}\}}{\sigma_{\underset{\sim}{X}}\sigma_{\underset{\sim}{Y}}}$$

APPLICATION EXAMPLE

- A company is selling a product *G* with different net profits corresponding to different levels of product sales

level of sales	*probability*	*net profits* [*M $*]
high	0.38	8
medium	0.12	4
low	0.50	*0*

- The standard deviation and variance of the net profits $\underset{\sim}{X}$ for the product are given by

APPLICATION EXAMPLE

$$E\{\underset{\sim}{X}\} = \sum_{i=1}^{n} x_i \, P\{\underset{\sim}{X} = x_i\} = 8(0.38) + 4(0.12) + 0(0.50)$$

$$= 3.52 \, M\$$$

$$var\{\underset{\sim}{X}\} = \sum_{i=1}^{n} \left[x_i - E\{\underset{\sim}{X}\} \right]^2 P\{\underset{\sim}{X} = x_i\}$$

$$= 0.38(8 - 3.52)^2 + 0.12(4 - 3.52)^2 + 0.5(0 - 3.52)^2$$

$$= 13.8496 \; (M\$)^2$$

$$\sigma_{\underset{\sim}{X}} = \sqrt{var\{\underset{\sim}{X}\}} = \sqrt{13.8496} = 3.72 \; M\$$$

ANOTHER EXAMPLE

- Consider the following probabilities:

$$P\{\underset{\sim}{Y} = 10 \mid \underset{\sim}{X} = 2\} = 0.9$$

$$P\{\underset{\sim}{X} = 2\} = 0.3 \qquad P\{\underset{\sim}{Y} = 20 \mid \underset{\sim}{X} = 2\} = 0.1$$

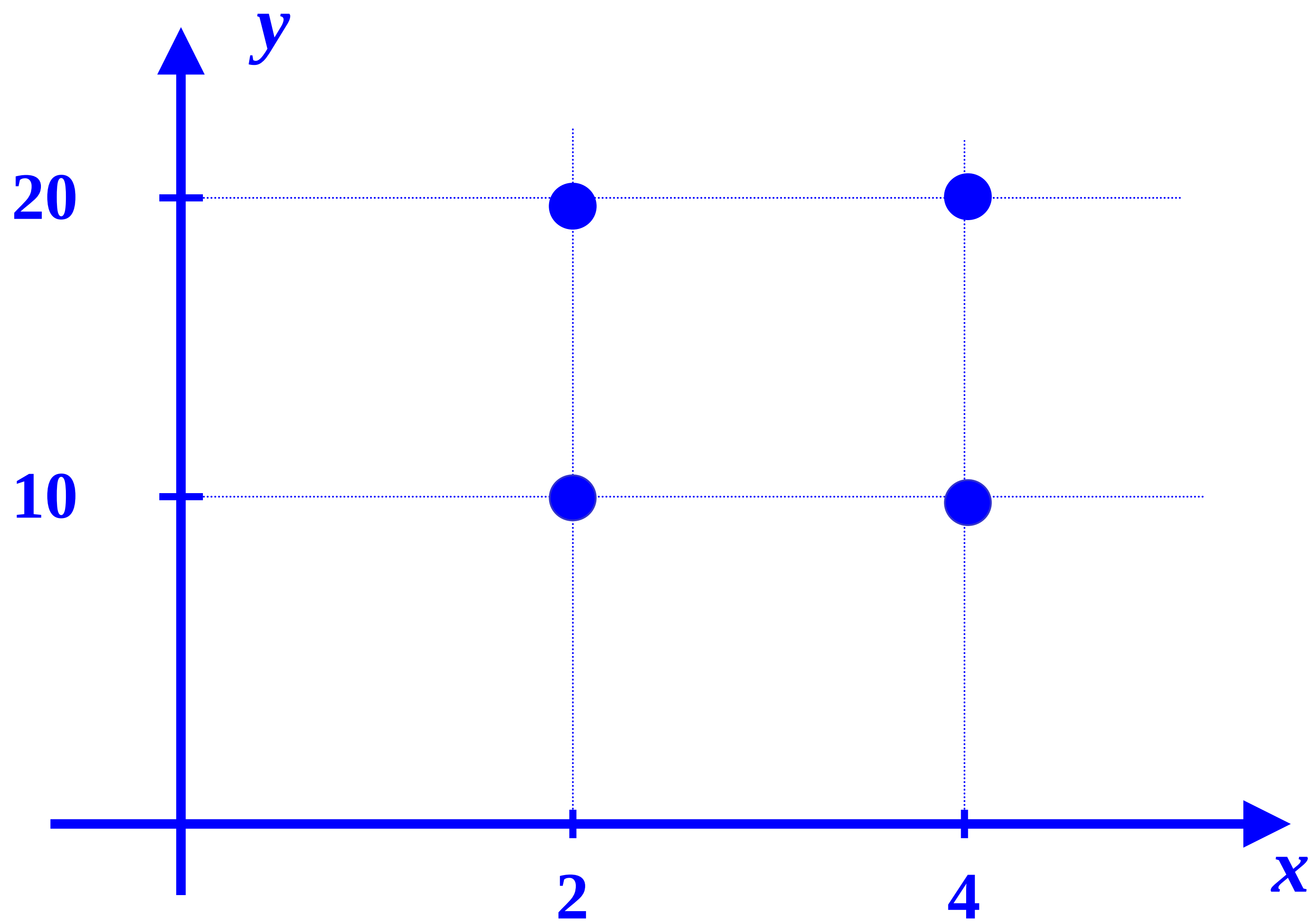
$$P\{\underset{\sim}{X} = 4\} = 0.7 \qquad P\{\underset{\sim}{Y} = 10 \mid \underset{\sim}{X} = 4\} = 0.25$$

$$P\{\underset{\sim}{Y} = 20 \mid \underset{\sim}{X} = 4\} = 0.75$$

and compute the covariance and correlation

between $\underset{\sim}{X}$ and $\underset{\sim}{Y}$

ANOTHER EXAMPLE



ANOTHER EXAMPLE

- Using the definition of conditional probability:

$$P\{\underset{\sim}{X}=2,\underset{\sim}{Y}=10\} = P\{\underset{\sim}{Y}=10 \mid \underset{\sim}{X}=2\}P\{\underset{\sim}{X}=2\}$$

$$= (0.9)(0.3) = 0.27$$

$$P\{\underset{\sim}{X}=2,\underset{\sim}{Y}=20\} = P\{\underset{\sim}{Y}=20 \mid \underset{\sim}{X}=2\}P\{\underset{\sim}{X}=2\}$$

$$= (0.1)(0.3) = 0.03$$

$$P\{\underset{\sim}{X}=4,\underset{\sim}{Y}=10\} = P\{\underset{\sim}{Y}=10 \mid \underset{\sim}{X}=4\}P\{\underset{\sim}{X}=4\}$$

$$= (0.25)(0.7) = 0.175$$

$$P\{\underset{\sim}{X}=4,\underset{\sim}{Y}=20\} = P\{\underset{\sim}{Y}=20 \mid \underset{\sim}{X}=4\}P\{\underset{\sim}{X}=4\}$$

$$= (0.75)(0.7) = 0.525$$

ANOTHER EXAMPLE

$$P\{\underset{\sim}{Y}=10\} = P\{\underset{\sim}{Y}=10 \mid \underset{\sim}{X}=2\}P\{\underset{\sim}{X}=2\} + P\{\underset{\sim}{Y}=10 \mid \underset{\sim}{X}=4\}P\{\underset{\sim}{X}=4\}$$

$$= 0.27 + 0.175 = 0.445$$

$$P\{\underset{\sim}{Y}=20\} = 1-(0.445) = 0.555$$

$$E\{\underset{\sim}{X}\} = (0.3)2 + (0.7)4 = 3.4$$

$$\sigma_{\underset{\sim}{X}} = \sqrt{(0.3)(-1.4)^2 + (0.7)(0.6)^2} = 0.917$$

$$E\{\underset{\sim}{Y}\} = (0.445)10 + (0.555)20 = 15.55$$

$$\sigma_{\underset{\sim}{Y}} = \sqrt{(0.445)(-4.45)^2 + (0.555)(14.45)^2} = 11.17$$

EXAMPLE

x_i	y_j	$x_i - E\{\underset{\sim}{X}\}$	$y_j - E\{\underset{\sim}{Y}\}$	$\left[x_i - E\{\underset{\sim}{X}\}\right] \cdot \left[y_j - E\{\underset{\sim}{Y}\}\right]$	$P\left\{ \underset{\sim}{X}, \underset{\sim}{Y} \Big\|_{x_i, y_i} \right\}$
2	**10**	**– 1.4**	**4.45**	**– 6.23**	**0.27**
2	**20**	**– 1.4**	**14.45**	**– 20.23**	**0.03**
4	**10**	**0.6**	**4.45**	**2.67**	**0.175**
4	**20**	**0.6**	**14.45**	**8.67**	**0.525**

EXAMPLE

$$cov\{\underset{\sim}{X}, \underset{\sim}{Y}\} = (0.27)(-6.23) + (0.03)(-20.23) + (0.175)2.67$$

$$= 2.73$$

$$\rho_{\underset{\sim}{X}\underset{\sim}{Y}} = \frac{cov\{\underset{\sim}{X}, \underset{\sim}{Y}\}}{\sigma_{\underset{\sim}{X}}\sigma_{\underset{\sim}{Y}}} = \frac{2.73}{(0.917)(4.970)} = 0.60$$

CONTINUOUS PROBABILITY DISTRIBUTIONS

- The *continuous probability distribution* specification of a continuous *r.v.* $\underset{\sim}{X}$ may be expressed either in terms of a
  - a *probability density function (p.d.f.)* $f_{\underset{\sim}{X}}(\cdot)$

$$f_{\underset{\sim}{X}}(x)\,dx \approx P\{x < \underset{\sim}{X} \leq x + dx\}$$

  - or, a *cumulative distribution function (c.d.f.)* $F_{\underset{\sim}{X}}(\cdot)$ which expresses the probability that the value of $\underset{\sim}{X}$ is less or equal to a given value x

$$F_{\underset{\sim}{X}}(x) = P\{\underset{\sim}{X} \leq x\} = \int_{-\infty}^{x} f_{\underset{\sim}{X}}(\xi)\,d\xi$$

EXPECTED VALUE, VARIANCE, STANDARD DEVIATION

- The *expected value* $\mu_{\underset{\sim}{X}}$ is given by

$$E\left\{\underset{\sim}{X}\right\}=\int_{-\infty}^{+\infty}\xi f_{\underset{\sim}{X}}(\xi)\,d\xi$$

- The *variance* $var\left\{\underset{\sim}{X}\right\}$ of $\underset{\sim}{X}$ is defined by

$$var\left\{\underset{\sim}{X}\right\}=\int_{-\infty}^{+\infty}\left[\xi-E\left\{\underset{\sim}{X}\right\}\right]^{2}f_{\underset{\sim}{X}}(\xi)\,d\xi$$

- The *standard deviation* $\sigma_{\underset{\sim}{X}}$ of $\underset{\sim}{X}$ is

$$\sigma_{\underset{\sim}{X}}=\sqrt{var\left\{\underset{\sim}{X}\right\}}$$

THE COVARIANCE AND THE CORRELATION

- The *covariance* $cov\left\{ \underset{\sim}{X}, \underset{\sim}{Y} \right\}$ of the two continuous *r.v.*s $\underset{\sim}{X}$ and $\underset{\sim}{Y}$

$$cov\left\{ \underset{\sim}{X}, \underset{\sim}{Y} \right\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[\xi - E\left\{ \underset{\sim}{X} \right\} \right] \left[\eta - E\left\{ \underset{\sim}{Y} \right\} \right] f_{\underset{\sim}{X},\underset{\sim}{Y}}\left(\xi, \eta \right) d\xi \, d\eta$$

where $f_{\underset{\sim}{X},\underset{\sim}{Y}}\left(\bullet, \bullet \right)$ is the joint density function of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$

- The *correlation coefficient* $\rho_{\underset{\sim}{X},\underset{\sim}{Y}}$ is computed by

$$\rho_{\underset{\sim}{X},\underset{\sim}{Y}} = \frac{cov\left\{ \underset{\sim}{X}, \underset{\sim}{Y} \right\}}{\sigma_{\underset{\sim}{X}} \sigma_{\underset{\sim}{Y}}}$$

APPLICATION

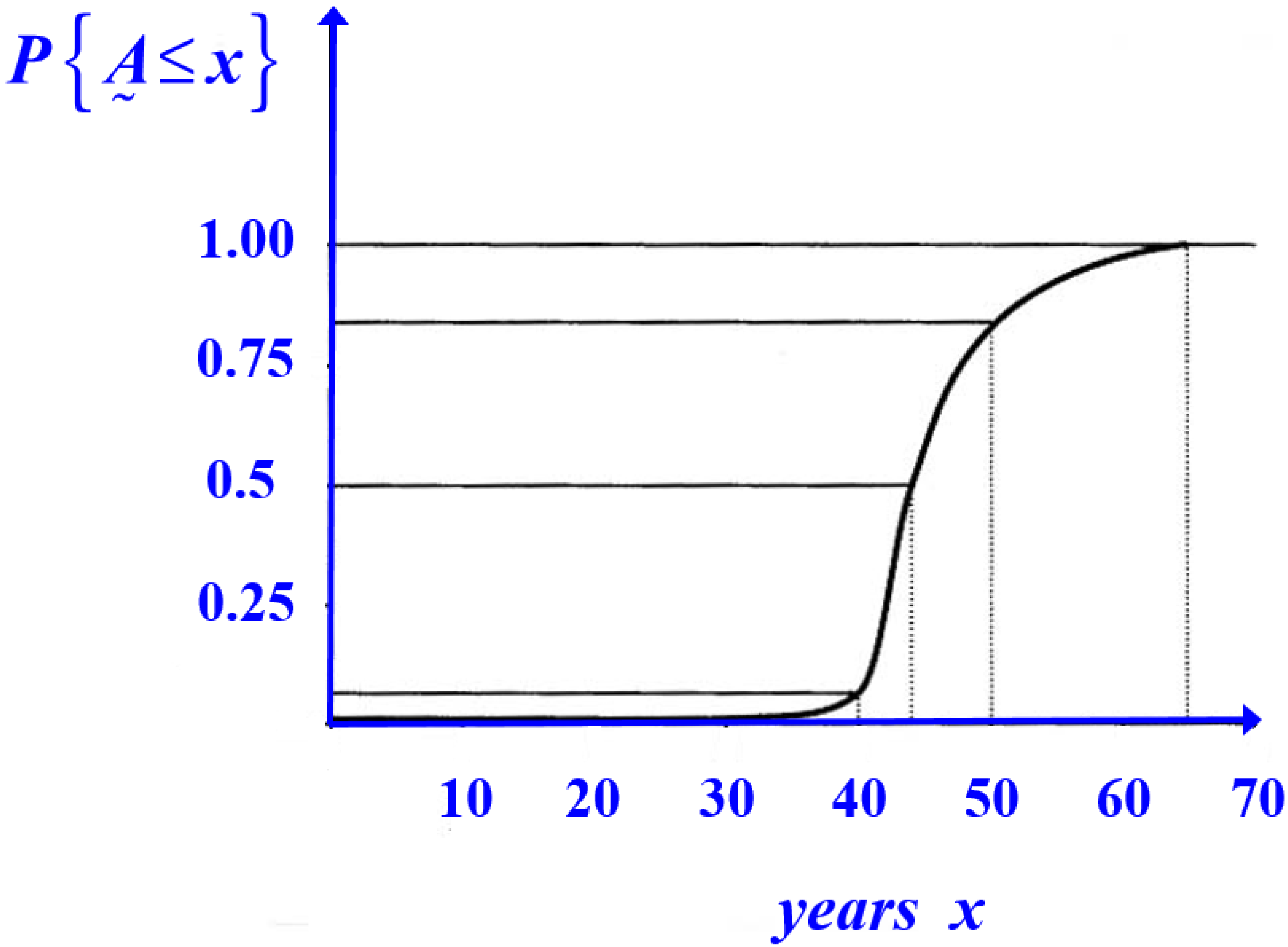
- We wish to guess the age $\underset{\sim}{A}$ of a movie star based on the following data:
  - we are sure that she is older than 29 and not older than 65
  - we assume the probability that she is between 40 and 50 is 0.8 and $P\left\{ \underset{\sim}{A} > 50 \right\} = 0.15$
  - we also estimate that $P\left\{ \underset{\sim}{A} \leq 40 \right\} = 0.05$ and $P\left\{ \underset{\sim}{A} \leq 44 \right\} = P\left\{ \underset{\sim}{A} > 44 \right\}$

APPLICATION

- We construct the table of cumulative probability

$P\left\{ \underset{\sim}{A} \leq 29 \right\} = 0.00$
$P\left\{ \underset{\sim}{A} \leq 40 \right\} = 0.05$
$P\left\{ \underset{\sim}{A} \leq 44 \right\} = 0.50$
$P\left\{ \underset{\sim}{A} \leq 50 \right\} = 0.85$
$P\left\{ \underset{\sim}{A} \leq 65 \right\} = 1.00$

APPLICATION



APPLICATION